Independence of Polarization
in Geometric Quantization

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1 Introduction

Classical (Hamiltonian) mechanics is formulated on symplectic manifolds (M, ω), on which observables are given by functions $f \in C^\infty(M)$. Systems are quantized by associating a Hilbert space $\mathcal{H}$ to M, and an (anti-)self-adjoint operator $\hat{f}$ on $\mathcal{H}$ to each function f.[1]

$\boxed{\text{Classical mechanics}}$		$\boxed{\text{Quantum mechanics}}$
(M, ω) : a symp. manifold	$\rightarrow$	$\mathcal{H}$: a Hilbert space
$f \in C^\infty(M)$	$\rightarrow$	$\hat{f} : \mathcal{H} \to \mathcal{H}$ (anti-)self-adjoint

We require that

- $f \mapsto \hat{f}$ is linear,
- $[\hat{f}, \hat{g}] = \widehat{\{f, g\}}$ for $f,\ g \in C^\infty(M)$,

where $\{\ ,\ \}$ is the Poisson bracket. In other words, $f \mapsto \hat{f}$ gives a Lie algebra homomorphism from $(C^\infty(M), \{\ ,\ \})$.

A basic example is the case of symplectic vector space $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with the standard symplectic form

$$\omega = \frac{1}{\hbar} \sum dp_i \wedge dq_i, \tag{1}$$

where $\hbar$ is the Planck's constant. In this case, we usually take the space $\mathcal{H} = L^2(\mathbb{R}^n)$ of L^2-functions in $q_1, \ldots q_n$, on which the positions q_i and momenta p_j are quantized by

$$\begin{aligned} \widehat{q_i}\varphi &= -\sqrt{-1} q_i \varphi\,, \\ \widehat{p_j}\varphi &= \hbar \frac{\partial}{\partial q_j} \varphi \end{aligned} \tag{2}$$

[1] In this note we consider anti-self-adjoint operators, because what we construct in the following are unitary representations.

for $\varphi \in \mathcal{H}$. Then we have the Heisenberg's uncertainty principle

$$[\widehat{q_i}, \widehat{p_j}] = \sqrt{-1}\hbar\delta_{ij} = \sqrt{-1}\{q_i, p_j\}.$$

This relation implies that q_i's, p_j's, together with a center $\mathbb{R}$ (*i.e.* constant functions) generate a Lie algebra heis($\mathbb{R}^{2n}$):

$$0 \longrightarrow \sqrt{-1}\mathbb{R} \longrightarrow \mathrm{heis}(\mathbb{R}^{2n}) \longrightarrow \mathbb{R}^{2n} \longrightarrow 0,$$

and (2) gives a representation of heis($\mathbb{R}^{2n}$). It is known that $\mathcal{H} = L^2(\mathbb{R}^n)$ is a (unique) irreducible representation of heis($\mathbb{R}^{2n}$).

There is another description of the irreducible representation of heis($\mathbb{R}^{2n}$), which is called the Bargmann-Fock representation. We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by $z_i = q_i + \sqrt{-1}p_i$. Then the representation is given by

$$\mathcal{H}' = \left\{ \psi : \mathbb{C}^n \to \mathbb{C} \ \text{ holomorphic} \,\middle|\, \int |\psi(z)|^2 e^{-|z|^2/2} \leq \infty \right\}$$

with

$$\begin{aligned} \widehat{z_i}\psi &= \sqrt{-1}z_i\psi, \\ \widehat{\bar{z}_i}\psi &= 2\sqrt{-1}\hbar\frac{\partial}{\partial z_i}\psi. \end{aligned}$$

The isomorphism $\mathcal{H} \to \mathcal{H}'$ is given by the Segal-Bargmann transform ([26, 27], [3, 4])

$$f(q) \longmapsto \int_{\mathbb{R}^n} A(z, q) f(q) dq,$$

where

$$A(z, q) = (2\pi)^{-3n/4} \exp\left(-\frac{1}{4}\sum({z_i}^2 + {q_i}^2) + \frac{1}{\sqrt{2}}\sum z_i q_i \right).$$

Geometric quantization is a generalization of these constructions. In geometric quantization, $\mathcal{H}$ and $\mathcal{H}'$ are called a *real quantization* and *Kähler quantization*, respectively. Similar equivalences are observed in several examples such as Abelian varieties and toric varieties. The case of Abelian varieties is understood in the theory of theta functions. Moreover, it is pointed out by A. Tyurin [31] that the equivalence is regarded as a part of mirror symmetry for Abelian varieties. On of the aim of this note is to exhibit the equivalences of real and Kähler quantizations through several examples.

Section 2 is an introduction to geometric quantization from the viewpoint of the equivalence of real and Kähler quantizations. We see the above mentioned equivalence for toric varieties, flag manifolds, and Abelian varieties in Section 3. In Section 4, we study this relation from the point of view of projective embeddings.

2 Geometric Quantization

2.1 Prequantization

Let (M,ω) be a symplectic manifold. We assume that $\frac{1}{2\pi}\omega$ represents an integral cohomology class:

$$\frac{1}{2\pi}[\omega] \in H^2(M,\mathbb{Z}).$$

Then there exists a *prequantum bundle* $(L,\nabla) \to M$, *i.e.* a complex line bundle with a unitary connection such that

$$c_1(L,\nabla) = \frac{1}{2\pi}\omega,$$

where $c_1(L,\nabla)$ is the first Chern form of (L,∇). We consider its automorphism group

$$\mathcal{G} = \mathrm{Aut}\,(L,\nabla) = \left\{ \begin{array}{ccc} L & \xrightarrow{\tilde{F}} & L \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & M \end{array} \;\middle|\; \begin{array}{c} \tilde{F} \text{ is unitary and} \\ \text{preserves } \nabla \end{array} \right\}.$$

From the definition, F preserves $\omega = 2\pi c_1(L,\nabla)$ for $(\tilde{F},F) \in \mathcal{G}$.

To see the Lie algebra $\mathrm{Lie}\,\mathcal{G}$ of $\mathcal{G}$, we first consider the group $\mathrm{Symp}(M,\omega)$ of symplectomorphims. Note that the space of vector fields on M is identified with the space $\Omega^1(M)$ of 1-forms by $\xi \mapsto i_\xi\omega$, where i_ξ is the contraction operator.

Lemma 2.1. *Under the above identification,*

$$\mathrm{Lie}\,\mathrm{Symp}(M,\omega) \cong \{\textit{closed 1-forms}\}.$$

Proof. This lemma follows from the closedness of ω and the formula

$$L_\xi\omega = i_\xi d\omega + d(i_\xi\omega)$$

for the Lie derivative. □

We define the *Hamiltonian vector field* ξ_f of $f \in C^\infty(M)$ by

$$i_{\xi_f}\omega = -df. \tag{3}$$

From Lemma 2.1, ξ_f preserves ω. A diffeomorphism F is said to be *Hamiltonian* if there exists a 1-parameter family of symplectomorphisms F_t with $F_0 = \mathrm{id}_M$, $F_1 = F$ such that $\frac{d}{dt}F_t = \xi_{f_t}$ is a Hamiltonian vector field for each t. We denote the group of Hamiltonian diffeomorphisms by $\mathrm{Ham}(M,\omega)$. Then, by definition, its Lie algebra is the space of exact 1-forms:

$$\mathrm{Lie}\,\mathrm{Ham}(M,\omega) \cong dC^\infty(M).$$

Remark 2.2. Recall that the Poisson bracket is given by

$$\{f, g\} = \omega(\xi_f, \xi_g) = \xi_f(g).$$

We choose the sign in (3) so that

$$[\xi_f, \xi_g] = \xi_{\{f,g\}}$$

holds.

Now we go back to $\mathrm{Lie}\,\mathcal{G}$. Note that $(\tilde{F}, F) \mapsto F$ gives a map $\mathcal{G} \to \mathrm{Symp}(M, \omega)$.

Proposition 2.3. $\mathrm{Lie}\,\mathcal{G} \cong (C^\infty(M), \{\ ,\ \})$*, and its action on the space* $\Gamma(M, L)$ *of smooth sections of* L *is given by*

$$s \longmapsto \hat{f}s := \nabla_{\xi_f} s + \sqrt{-1} f s \tag{4}$$

for $f \in C^\infty(M)$*.*

In fact, $\mathcal{G}$ is a central extension

$$1 \longrightarrow S^1 \longrightarrow \mathcal{G} \longrightarrow \mathrm{Ham}\,(M, \omega) \longrightarrow 1$$

of $\mathrm{Ham}(M, \omega)$, corresponding to the natural exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \longrightarrow d\,C^\infty(M) \longrightarrow 0.$$

Proof. Let g_t be a 1-parameter family in $\mathcal{G}$ such that $g_0 = \mathrm{id}$. Then its infinitesimal action $\frac{d}{dt}\big|_{t=0} g_t$ has the form $\nabla_\xi + \sqrt{-1} f$ for some $\xi \in \mathrm{Lie}\,\mathrm{Symp}(M, \omega)$ and $f \in C^\infty(M)$. The proposition follows from

$$\left.\frac{d}{dt}\right|_{t=0} g_t^* \nabla = \sqrt{-1}(df - i_\xi \omega).$$

□

Hence $\Gamma(M, L)$ gives a representation of the Lie algebra $(C^\infty(M), \{\ ,\ \})$. However this is not our goal, because this space is "too large". Recall that, in the case of $M = \mathbb{R}^{2n}$, $\mathcal{H} = L^2(\mathbb{R}^n)$ consists of functions depending only on the q-variables. In the next subsection, we introduce the notion of polarization to eliminate a half of the variables.

Remark 2.4. When we replace L with its tensor power L^k, then the symplectic form $\omega = 2\pi c_1(L, \nabla)$ is replaced by $k\omega = 2\pi c_1(L^k, \nabla)$. Comparing this with (1), $1/k$ can be regarded as the Planck's constant $\hbar$, and the limit $k \to \infty$ corresponds to the semiclassical limit. This limit plays important roles in algebraic and symplectic geometry (see [9]).

2.2 Polarizations

Definition 2.5. A *polarization* P is an integrable Lagrangian distribution in the complexified tangent bundle $TM \otimes \mathbb{C}$, *i.e.* $[P,P] \subset P$, $\mathrm{rank}_{\mathbb{C}} P = n$, and $\omega|_P = 0$, where ω is extended to $TM \otimes \mathbb{C}$ complex bilinearly.

We define the space of polarized sections by

$$\Gamma_P(M,L) := \{ s \in \Gamma(M,L) \,|\, \nabla_\xi s = 0 \text{ for all } \xi \in P \}.$$

We first check the integrability condition for the equation $\nabla_\xi s = 0$. Since the curvature of ∇ coincides with $-\sqrt{-1}\omega$, we have

$$[\nabla_\xi, \nabla_\eta] s = (\nabla_{[\xi,\eta]} - \sqrt{-1}\omega(\xi,\eta))s$$

From the integrability condition for P, we have $[\xi,\eta] \in P$ for $\xi, \eta \in P$. On the other hand, the second term of the right hand side vanishes from the Lagrangian condition.

There are two important classes of polarizations.

Real polarization Let $\pi : M \to B$ be a Lagrangian fibration, namely, its general fibers are Lagrangian submanifolds.[2] Then the complexified relative tangent bundle

$$P = T_{M/B} \otimes \mathbb{C} = \ker(d\pi : TM \to TB) \otimes \mathbb{C}$$

is a polarization. The corresponding vector space consists of sections which are covariantly constant along the Lagrangian fibers.

Example 2.6 (symplectic vector space)**.** Let $M = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ with $\omega = \sum dp_i \wedge dq^i$. In this case, a trivial bundle with $\nabla = d + \sqrt{-1} q^i dp_i$ gives a prequantum bundle on M. Since the group $\mathbb{R}^{2n}$ of translations on M is a subgroup of $\mathrm{Ham}(M,\omega)$, it lifts to a subgroup $\mathrm{Heis}(\mathbb{R}^{2n})$ of $\mathcal{G}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{Ham}(M,\omega) & \longrightarrow & 0 \\
 & & \| & & \cup & & \cup & & \\
0 & \longrightarrow & S^1 & \longrightarrow & \mathrm{Heis}(\mathbb{R}^{2n}) & \longrightarrow & \mathbb{R}^{2n} & \longrightarrow & 0.
\end{array}$$

$\mathrm{Heis}(\mathbb{R}^{2n})$ is called the *Heisenberg group.* Note that the Lie algebra of $\mathrm{Heis}(\mathbb{R}^{2n})$ is $\mathrm{heis}(\mathbb{R}^{2n})$ in Section 1.

The natural projection $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is a Lagrangian fibration. In this case, the corresponding polarized sections can be identified with functions depending only on q-variables. Note that the action of $\mathbb{R}^{2n}$ preserves the Lagrangian fibration and hence $\mathrm{Heis}(\mathbb{R}^{2n})$ acts on the space of polarized sections. After taking a completion, we obtain the Heisenberg representation $L^2(\mathbb{R}^n)$.

We need to modify the definition of the real quantization for general cases. This is discussed in the next subsection.

[2] We allow Lagrangian fibrations to have degenerate fibers.

Kähler polarization Assume that (M, ω) is a Kähler manifold. Then the anti-holomorphic tangent bundle

$$P = T^{0,1}M \subset TM \otimes \mathbb{C}$$

gives another polarization: The integrability condition for P is equivalent to the integrability of the complex structure, and the Lagrangian condition follows from the fact that ω is a $(1,1)$-form. In this case, L is a holomorphic line bundle since its curvature $-\sqrt{-1}\omega$ is of type (1,1). Then the polarized condition becomes $\overline{\partial} s = 0$, which means that the space of polarized sections is nothing but the space of holomorphic sections:

$$\Gamma_P(M, L) = H^0(M, L).$$

Thanks to results from algebraic geometry, we can see its dimension using Riemann-Roch theorem, and the dependence of $\Gamma_P(M, L)$ on the complex structures on M (see [18]).

Example 2.7 (symplectic vector space)**.** We consider the case of symplectic vector space again. We identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Then L is a trivial holomorphic line bundle with a Hermitian metric $e^{-|z|^2/2}$. The corresponding vector space is the space of holomorphic sections with finite L^2-norms with respect to the Hermitian metric $e^{-|z|^2/2}$. Since $\mathrm{Heis}(\mathbb{R}^{2n})$ preserves the complex structure, we obtain a representation of the Heisenberg group. This is an irreducible representation which is called the *Bargmann-Fock representation*, and isomorphic to the real quantization $L^2(\mathbb{R}^n)$ as mentioned in Section 1.

Remark 2.8. Hall [17] proved the equivalence of real and Kähler quantizations on the cotangent bundles of compact Lie groups.

Remark 2.9. Unfortunately, from the viewpoint of geometric quantization, the subgroup of $\mathcal{G}$ which preserves a polarization is small (may be trivial) in general. On the other hand, this fact enables us to construct a "good" moduli space of polarized[3] algebraic varieties.

2.3 Bohr-Sommefeld condition

We assume that M is compact and admits a Lagrangian fibration $\pi : M \to B$. Then π is locally given by a completely integrable system, and in particular, general fibers of π are Lagrangian tori (see [10]). In this section, we consider only the smooth part of the fibration. By definition of a prequantum bundle, the restriction $(L, \nabla)|_{\pi^{-1}(b)}$ of L to each Lagrangian fiber is flat:

$$c_1(L, \nabla)|_{\pi^{-1}(b)} = 2\pi\omega|_{\pi^{-1}(b)} = 0.$$

However, it is not trivial in general, since the fiber has a non-trivial fundamental group $\pi_1(\pi^{-1}(b)) = \mathbb{Z}^n$. This implies that there is no nontrivial polarized section for the real polarization $P = \ker \pi$. We modify the definition of polarized sections in the following way.

[3]in the sense of algebraic geometry

Definition 2.10. A fiber $\pi^{-1}(b)$ is called a *Bohr-Sommerfeld fiber* if the restriction $(L, \nabla)|_{\pi^{-1}(b)}$ is trivial.

By considering sections supported only on Bohr-Sommerfeld fibers, we have non-trivial vector spaces. We can introduce the "Planck's constant" $1/k$.

Definition 2.11. A fiber $\pi^{-1}(b)$ is called a *Bohr-Sommerfeld fiber of level k* (or k-BS for short) if the restriction $(L^k, \nabla)|_{\pi^{-1}(b)}$ is trivial.

The space of polarized sections is defied by

$$\Gamma_P(M, L^k) := \{s \mid \operatorname{supp} s \subset k\text{-BS fibers},\ \nabla_\xi s = 0,\ \xi \in P\}.$$

We remark that k-BS fibers appear discretely. This is easily seen using *action-angle variables*. On a neighborhood of a smooth torus fiber, we can take a coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ satisfying

- $\omega = \sum dx^i \wedge dy^i$,
- $0 \le x^i < 1$ are coordinates on the torus fibers, and
- y^i's are coordinates on the base space such that $e^{2\pi\sqrt{-1}y^i}$ gives the holonomy of (L, ∇) along the loop corresponding to the x^i-axis.

Then $\pi^{-1}(y)$ satisfies the k-BS condition if and only if its action coordinates y^i's take values in $\frac{1}{k}\mathbb{Z}$. In particular, we have

$$\dim \Gamma_P(M, L^k) = \text{the number of } k\text{-BS fibers}$$

(if degenerate fibers do not contribute badly).

Remark 2.12. Śniatycki [29] gives a cohomological definition for real quantizations and proves the equivalence with the above definition.

3 Examples

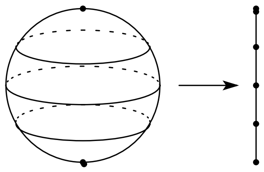
3.1 Toric varieties

For simplicity, we consider the case of $M = \mathbb{CP}^1$ with the Fubini-Study metric $\omega = \omega_{\mathrm{FS}}$ and the hyperplane bundle $L = \mathcal{O}(1)$. As a Lagrangian fibration, we consider the moment map

$$\mu : \mathbb{CP}^1 \longrightarrow [0, 1], \quad [z_0 : z_1] \longmapsto \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}$$

of a natural S^1-action, where $[z_0 : z_1]$ is a homogeneous coordinate on $\mathbb{CP}^1$. Then a fiber $\mu^{-1}(b)$ satisfies the k-BS condition if and only if $b \in \frac{1}{k}\mathbb{Z} \cap [0, 1]$. In particular,

$$\dim \Gamma_{\ker d\mu}(\mathbb{CP}^1, \mathcal{O}(k)) = \# \frac{1}{k}\mathbb{Z} \cap [0, 1] = k + 1.$$



On the other hand, the Kähler quantization $H^0(\mathbb{CP}^1, \mathcal{O}(k))$ has a monomial basis

$$z_0^k,\ kz_0^{k-1}z_1, \ldots, \binom{k}{i} z_0^{k-i} z_1^i, \ldots, z_1^k. \tag{5}$$

Note that also these monomials are indexed by lattice points in the moment polytope $[0,1]$:

$$\binom{k}{i} z_0^{k-i} z_1^i \longleftrightarrow \frac{i}{k} \in [0,1] \cap \frac{1}{k}\mathbb{Z}.$$

In particular, two quantization are equivalent (as an S^1-representation):

$$\Gamma_{\ker d\mu}(\mathbb{CP}^1, \mathcal{O}(k)) \cong H^0(\mathbb{CP}^1, \mathcal{O}(k)).$$

3.2 Flag manifolds

Let $M = U(n)/T$ be a complex flag manifold, where T is a maximal torus of $U(n)$ consists of diagonal matrices. We fix $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i \in \mathbb{Z}$ and $\lambda_1 > \cdots > \lambda_n$. Then λ gives a character $T \to \mathbb{C}^*$ and hence defines a holomorphic line bundle

$$L_\lambda = (U(n) \times \mathbb{C})/T \longrightarrow M = U(n)/T.$$

Since L_λ is ample, we can take a (unique) $U(n)$-invariant Kähler form ω_λ in the class $c_1(L_\lambda)$. Then $U(n)$ acts holomorphically on L_λ (*i.e.* $U(n) \subset \mathcal{G}$), and we have a representation $H^0(M, L_\lambda)$ of $U(n)$.[4]

Theorem 3.1 (Borel-Weil)**.** *$H^i(M, L_\lambda) = 0$ for $i \neq 0$ and $H^0(M, L_\lambda)$ is an irreducible representation of $U(n)$ of highest weight λ. Furthermore, every irreducible representation is given in this way.*

Next we consider a real quantization. We identify M with the (co)adjoint orbit $\mathcal{O}_\lambda$ of λ, *i.e.* the space of Hermitian matrices with fixed eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$U(n)/T \longleftrightarrow \mathcal{O}_\lambda, \quad gT \longleftrightarrow g\lambda g^*.$$

Then ω_λ coincides with the Kostant-Kirillov form under this identification. For each $x \in \mathcal{O}_\lambda$, we denote its $i \times i$ upper-left submatrix by $x^{(i)}$. Since every $x^{(i)}$ is Hermitian, it has real eigenvalues $\lambda_1^{(i)} \geq \cdots \geq \lambda_i^{(i)}$. It is easy to see that these

[4] Since $L_\lambda^k = L_{k\lambda}$, we restrict ourselves to the case $k = 1$.

eigenvalues satisfy

$$
\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-1} & & \lambda_n \\
& \rotatebox{-45}{$\geq$} \quad \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} \quad \rotatebox{45}{$\leq$} & & & & \rotatebox{-45}{$\geq$} \quad \rotatebox{45}{$\leq$} & \\
& \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & & & \lambda_{n-1}^{(n-1)} & \\
& & \rotatebox{-45}{$\geq$} \quad \rotatebox{45}{$\leq$} & & & & \rotatebox{45}{$\leq$} & & \\
& & \lambda_1^{(n-2)} & & & & \lambda_{n-2}^{(n-2)} & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & \ddots & & \iddots & & & \\
& & & & \rotatebox{-45}{$\geq$} \quad \rotatebox{45}{$\leq$} & & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
\tag{6}
$$

By associating the collection $\left(\lambda_j^{(i)}\right)_{1 \leq j \leq i \leq n-1}$ of eigenvalues to each $x \in \mathcal{O}_\lambda$, we obtain a completely integrable system $\pi : M \to \mathbb{R}^{n(n-1)/2}$. This is called the *Gelfand-Cetlin system*. Its image $P_\lambda = \pi(M) \subset \mathbb{R}^{n(n-1)/2}$, the *Gelfand-Cetlin polytope*, is a polytope consists of points satisfying (6).

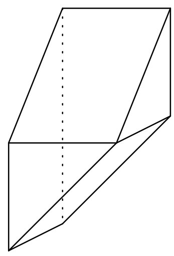


Figure 1: The Gelfand-Cetlin polytope for $n = 3$

Theorem 3.2 (Guillemin-Sternberg [16])**.** *A fiber $\pi^{-1}(b)$ satisfies the Bohr-Sommerfeld condition if and only if $b \in P_\lambda \cap \mathbb{Z}^{n(n-1)/2}$.*

It is known that the Kähler quantization $H^0(M.L_\lambda)$ has a basis indexed by the lattice points $P_\mathbb{Z} = P_\lambda \cap \mathbb{Z}^{n(n-1)/2}$ of P_λ ([13]). For each $i = 1, \dots, n-1$, we identify $U(i)$ with a subgroup of $U(n)$ of the form

$$
\left(\begin{array}{c|c} 1_{n-i} & 0 \\ \hline 0 & U(i) \end{array} \right) \subset U(n).
$$

Then we have an irreducible decomposition $H^0(M, L_\lambda) = \bigoplus V_\mu$ as a $U(n-1)$-representation, where V_μ is the irreducible representation of $U(n-1)$ of highest weight $\mu = (\mu_1 \geq \cdots \geq \mu_{n-1})$. It is known that

- the multiplicity of V_μ is at most 1 for each μ,

- V_μ is an irreducible component of $H^0(M, L_\lambda)$ if and only if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

(see, for example, [35]). Repeating this step successively, we obtain an irreducible decomposition

$$H^0(X, L_\lambda) = \bigoplus_{\Lambda \in P_{\mathbb{Z}}} V_\Lambda, \quad V_\Lambda \cong \mathbb{C} \tag{7}$$

as a $U(1)$-representation. We call (7) a *Gelfand-Cetlin decomposition.* A *Gelfand-Cetlin basis* is given by choosing a basis of V_Λ for each $\Lambda \in P_{\mathbb{Z}}$. In particular, we have an isomorphism

$$\Gamma_{\ker d\pi}(M, L_\lambda) \cong H^0(M, L_\lambda)$$

as a vector space.

Remark 3.3. Note that the $U(n)$-action on M does not preserve the Gelfand-Cetlin system, and hence $U(n)$ does not acts on the real quantization $\Gamma_{\ker d\pi}(M, L_\lambda)$.

3.3 Abelian varieties

Let $A = \mathbb{C}^n/\Omega\mathbb{Z}^n + \mathbb{Z}^n$ be an Abelian variety with a Kähler form

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum g_{ij} dz^i \wedge d\bar{z}^j = -\sum dx^i \wedge dy^i, \tag{8}$$

where Ω is an $n \times n$ symmetric matrix with positive definite imaginary part $\operatorname{Im}\Omega = (g_{ij})^{-1}$, and $z = \Omega x + y$. We take an ample line bundle L of degree 1 defined by

$$L = (\mathbb{C}^n \times \mathbb{C})/\sim,$$

where

$$(z, \zeta) \sim (z + \lambda, e^{2\pi {}^t\lambda (\operatorname{Im}\Omega)^{-1} z + \pi {}^t\lambda (\operatorname{Im}\Omega)^{-1}\lambda} \zeta)$$

for $\lambda \in \Omega\mathbb{Z}^n + \mathbb{Z}^n$. Then $\omega = c_1(L, h_0)$ for some Hermitian metric h_0 on L (such h_0 is unique up to constant multiples).

Remark 3.4. The choice of L is not essential. In fact, any other principal polarization can be obtained as a pull-back of L by some translation on A. We remark that L is *symmetric*: $(-1)^*L \cong L$, where (-1) is the inverse morphism of A. This property is necessary when we consider the case of Kummer varieties.

Definition 3.5. Let A_k be the subgroup of k-torsion points in A. We define

$$G_k = \left\{ \begin{array}{ccc|c} L^k & \xrightarrow{F} & L^k & w \in A_k, \\ \downarrow & & \downarrow & \\ A & \xrightarrow{\tau_w} & A & F \text{ is unitary} \end{array} \right\} \subset \mathcal{G}_k = \operatorname{Aut}(A, L^k),$$

where $\tau_w : z \mapsto z + w$ is the translation by w. G_k is obtained as an central extension of A_k

$$1 \longrightarrow S^1 \longrightarrow G_k \longrightarrow A_k \longrightarrow 0.$$

Note that A_k is the maximal subgroup of translations which can be lifted to isomorphisms of L^k:

$$\tau_w^* L^k \cong L^k \iff w \in A_k.$$

Theorem 3.6 ([20] Proposition 3.2)**.** *$H^0(X, L^k)$ is an irreducible representation of the Heisenberg group G_k (which is unique up to isomorphisms).*

Holomorphic sections of L^k are given by *theta functions.* Let T^n be an n-dimensional tours $\mathbb{R}^n/\mathbb{Z}^n$, and denote the subgroup of k-torsion points in T^n by

$$T_k^n = \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n = \{b_i\}_{i=1,\ldots,k^n} \subset T^n\,.$$

Then the collection of

$$s_{b_i}(z) = s_i(z) = Ck^{-\frac{n}{4}} \exp\left(\frac{\pi}{2}k\,{}^t z(\operatorname{Im}\Omega)z\right) \vartheta \begin{bmatrix} 0 \\ -b_i \end{bmatrix} (k^{-1}\Omega, z)\,, \quad i = 1, \ldots, k^n,$$

gives an *orthonormal basis* of $H^0(A, L^k)$ with respect to the L^2-inner product, where

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega, z) = \sum_{l \in \mathbb{Z}^n} \exp\Big(\pi\sqrt{-1}\,{}^t(l+a)\Omega(l+a) + 2\pi\sqrt{-1}\,{}^t(l+a)(z+b)\Big)\,,$$

and C is a constant depends only on Ω (and h_0).

We next consider a real quantization. Here we take the following Lagrangian fibration

$$\pi : (A, \omega_0) \longrightarrow T^n, \quad \Omega x + y \longmapsto y.$$

Proposition 3.7 (Weitsman [33])**.** *A fiber $\pi^{-1}(b)$ of π satisfies the Bohr-Sommerfeld condition of level k if and only if $b \in T_k^n = \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$.*

For each $b_i \in T_k^n$, we take a covariantly constant section σ_i of $(L^k, \nabla)|_{\pi^{-1}(b_i)}$ with $\|\sigma_i\|_{L^2} = 1$. Then the real quantization is given by

$$\Gamma_{\ker d\pi}(A, L^k) = \bigoplus_{b_i \in T_k^n} \mathbb{C}\sigma_i.$$

In particular, we have an isomorphism

$$\bigoplus_{b_i \in T_k^b} \mathbb{C}\sigma_i \cong H^0(A, L^k), \quad \sigma_i \longleftrightarrow s_i. \tag{9}$$

This isomorphism is realized by using the *Bergman kernel.* The Bergman kernel $\Pi_k(z, w)$ is the integral kernel of the orthogonal projection

$$\Pi_k : L^2(A, L^k) \longrightarrow H^0(A, L^k)$$

from the space of L^2-sections to the space of holomorphic sections. From the fact that $\{s_i\}$ is an orthonormal, the Bergman kernel is given by

$$\Pi_k(z, w) = \sum_{i=1}^{k^n} s_i(z) s_i(w)^*.$$

We extend Π_k to the real quantization. Then we have

$$s_i = \left(\frac{k}{\pi}\right)^{-n/4} \Pi_k(\sigma_i) = \left(\frac{k}{\pi}\right)^{-n/4} \int_{\pi^{-1}(b)} \Pi_k(z,x)\sigma_i(x)dx. \qquad (10)$$

This is a special case of the "BPU construction" [6].

Mirror symmetry The isomorphism (9) can be understood also in terms of *mirror symmetry* for Abelian varieties. Mathematically, mirror symmetry is a conjectural duality between complex geometry on a Kähler manifold M and symplectic geometry on another Kähler manifold W which is called a mirror partner of M. We regard $H^0(A, L^k)$ as an object in the complex side. The corresponding object in the symplectic side is a Floer homology of Lagrangian intersections. Roughly speaking, chains of a Floer homology are generated by intersection points of Lagrangian submanifolds, and the boundary operator is given by counting holomorphic disks whose boundaries are on the Lagrangian submanifolds.

According to the SYZ conjecture [30], a mirror partner $\hat{A}$ of A is given by dualizing the torus fibers of $\pi : A \to T^n$. Since the dual torus parametrizes flat line bundles on a torus fiber of π, $b \mapsto \left[L^k|_{\pi^{-1}(b)}\right]$ determines a Lagrangian section S_k of the dual torus fibration $\hat{\pi} : \hat{A} \to T^n$. In particular, the trivial bundle $\mathcal{O}_A = L^0$ corresponds to the zero section S_0. Then $\pi^{-1}(b)$ is a k-BS fiber if and only if $b \in S_0 \cap S_k$, where we identify T^n with S_0. The following theorem is a part of the mirror symmetry for Abelian varieties:

Theorem 3.8 (Polischuk-Zaslow [23], Fukaya [12])**.** *The Floer homology $HF(S_0, S_k)$ of S_0 and S_k is given by*

$$HF(S_0, S_k) = \bigoplus_{b \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n} \mathbb{C}[b],$$

and $s_b \mapsto [b]$ gives an isomorphism

$$H^0(A, L^k) = \mathrm{Hom}\,(L^0, L^k) \cong HF(S_0, S_k).$$

Remark 3.9. Gross [14, 15] and Tyurin [31] obtained a similar result for K3 surfaces by using mirror symmetry given in terms of the K3 lattice ([8]). In this case, we can write down explicitly the homology classes of the Lagrangian sections corresponding to prequantum bundles, and hence can calculate the number of their intersection points.

Using a similar idea, Andersen proved the following.

Theorem 3.10 (Andersen [1])**.** *Let (M, ω) be a compact Kähler manifold of complex dimension n, $L \to M$ a prequantum bundle. Assume that M admits a Lagrangian fibration $\pi : M \to B$ with no degenerate fiber. Then*

$$\dim H^0(M, L^k) = \dim \Gamma_{T_{M/B}}(L^k)$$

for large k.

Proof. From the Riemann-Roch theorem and a vanishing theorem, we have

$$\dim H^0(M, L^k) = \int_M ch(L^k)\hat{A}(TM) = \int_M \exp(k\omega)\hat{A}(TM)\,.$$

Since $\pi : M \to B$ has no singular fiber, there exists a $\mathbb{Z}^n$-bundle $\Lambda \subset T^*B$ ($\Lambda = R^1\pi_*\mathbb{Z}$) such that the Lagrangian fibration is locally isomorphic to the natural projection $T^*B/\Lambda \to B$ (see [10]). We consider the following exact sequence

$$0 \longrightarrow T_{M/B} \longrightarrow TM \longrightarrow \pi^*TB \longrightarrow 0.$$

By using the symplectic form, $T_{M/B}$ can be identified with π^*T^*B. Since T^*B contains a lattice bundle Λ of maximal rank, we have

$$\hat{A}(TM) = \hat{A}(T_{M/B})\hat{A}(\pi^*TB) = \pi^*\big(\hat{A}(T^*B)\hat{A}(TB)\big) = 1\,.$$

Consequently we have

$$\dim H^0(M, L^k) = \int_M k^n\omega^n/n! = k^n \cdot \mathrm{vol}(M,\omega)\,.$$

Next we calculate the number of Bohr-Sommerfeld fibers. We define a dual torus fibration of $\pi : M \to B$ by $\check{\pi} : TB/\Lambda^* \to B$, where $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset TB$ is the dual lattice of Λ. Let $S_k : B \to TB/\Lambda^*$ be a section given by $b \mapsto [L^k|_{\pi^{-1}(b)}]$. Then

$$\dim \Gamma_{T_{M/B}}(L^k) = \#(S_0(B) \cap S_k(B))\,.$$

S_k can be written explicitly as follows. Let $(x^1, \dots, x^n, y_1, \dots, y_n)$ be the action-angle coordinate, and take the dual coordinate $(v^1, \dots, v^n)$ of $(y_1, \dots, y_n)$. Then S_k is given by

$$S_k(x^1, \dots, x^n) = (x^1, \dots, x^n, kx^1, \dots, kx^n) \tag{11}$$

with respect to this coordinate. This implies that that S_k and S_0 intersect transversely and positively (under a suitable orientation). Therefore the number of Bohr-Sommerfeld fibers of level k coincides with the intersection number of S_k and S_0. Since $\alpha = dv^1 \wedge \cdots \wedge dv^n \in \Omega^n(TB/\Lambda^*)$. gives the Poincaré dual of $\lambda_0(B)$, we have

$$\begin{aligned}\dim \Gamma_{T_{M/B}}(L^k) &= \int_{S_k(B)\cap S_0(B)} 1 \\ &= \int_{S_k(B)} \alpha = \int_B S_k^*\alpha\,.\end{aligned}$$

By using

$$S_k^*\alpha = k^n dx^1 \wedge \cdots \wedge dx^n = k^n \pi_*\omega^n/n!,$$

which follows from (11), we obtain

$$\begin{aligned}\dim \Gamma_{T_{M/B}}(L^k) &= k^n \int_B \pi_* \omega^n / n! \\ &= k^n \int_M \omega^n / n! = k^n \cdot \mathrm{vol}(M, \omega)\,.\end{aligned}$$

□

3.4 Moduli of vector bundles

Let $\mathcal{M}$ be a moduli space of vector bundles of fixed rank and degree on a compact Riemann surface. $\mathcal{M}$ admits a Kähler structure with a prequantum line bundle $\mathcal{L} \to \mathcal{M}$ (see [2]). Holomorphic sections of $\mathcal{L}$ are called *generalized theta functions*. The quantization for $(\mathcal{M}, \mathcal{L})$ is related to many areas in mathematics and mathematical physics.

Moreover, $\mathcal{M}$ admits a Lagrangian fibration. Roughly speaking, it is constructed in the following way. We decompose the Riemann surface into pairs of pants. Then, to twist each vector bundle along the boundaries of the components defines a torus action on an open dense subset of $\mathcal{M}$, which extends to a Lagrangian fibration of $\mathcal{M}$.

Theorem 3.11 (Jeffrey-Weitsman [19], A. Tyurin [32])**.** *For $\mathcal{M}$, the real and Kähler quantization are equivalent.*

4 Projective Embeddings and Lagrangian Fibrations

We have seen that real and Kähler quantizations are equivalent in several examples. In particular, in each compact case, the canonical basis of the real quantization gives a basis of the Kähler quantization $H^0(M, L^k)$ via the isomorphism. It is natural to ask to what extent such basis have information of Lagrangian fibrations. We study the relation through projective embeddings.

Example 4.1 (Toric varieties)**.** Recall that the S^1-action on $\mathbb{CP}^1$ gives a holomorphic automorphism of $\mathcal{O}(1)$ (*i.e.* $S^1 \subset \mathcal{G}$), and monomials are weight vectors of the S^1-action on the Kähler quantization $H^0(\mathbb{CP}^1, \mathcal{O}(k))$. In particular, the Lagrangian fibration is recovered from the monomial basis. Let

$$\iota_k : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^k, \quad z \longmapsto \left[z_0^k : \cdots : \binom{k}{i} z_0^{k-i} z_1^i : \cdots : z_1^k \right].$$

be a projective embedding given by the monomial basis (5), and

$$\mu_k : \mathbb{CP}^k \longrightarrow \Delta_k \subset \mathrm{Lie}(T^k)^*$$

the moment map of a natural T^k-action, where Δ_k is the moment polytope of $\mathbb{CP}^k$. Since ι_k is S^1-equivariant, the restriction $\pi_k = \mu_k \circ \iota_k : \mathbb{CP}^1 \to \Delta_k$ of μ_k to $\mathbb{CP}^1$ is also a moment map of the S^1-action on $\mathbb{CP}^1$.

However, the situation is not so simple in other examples. We consider the case of Abelian varieties and Kummer varieties in detail.

4.1 The case of Abelian varieties

We used the notation in 3.3. As remarked above, the T^n-action of translations in the fiber direction of $\pi : A \to T^n$ does not lift to L^k. Thus s_i does not have the symmetry of this T^n-action, and hence the Lagrangian fibration cannot be recovered from this basis $\{s_{b_i}\}$ for fixed k. On the other hand, each s_b becomes concentrated on the fiber $\pi^{-1}(b)$ over $b \in T^n$ as $k \to \infty$. More precisely,

Lemma 4.2. *There exist constants $C, c > 0$ independent of k such that*

$$|s_i(z)|^2_{h_0} \leq Ck^{n/2} e^{-ck \cdot d(y,b_i)^2}$$

for each $z = \Omega x + y \in A$, where $d(\ ,\ )$ is a distance on T^n .

It is natural from this fact to expect that the sequence of the basis $\{s_b\}$ reconstructs the Lagrangian fibration in the limit $k \to \infty$.

We consider a projective embedding

$$\iota_k : A \hookrightarrow \mathbb{CP}^{k^n-1}, \quad z \mapsto \left[\vartheta \begin{bmatrix} 0 \\ -b_1 \end{bmatrix} (k^{-1}\Omega, z) : \cdots : \vartheta \begin{bmatrix} 0 \\ -b_{k^n} \end{bmatrix} (k^{-1}\Omega, z)\right]$$

given by $\{s_b\}_{b \in T^n_k}$, and restrict the moment map of the torus action μ_k to A:

$$\pi_k := \mu_k \circ \iota_k : A \to B_k, \quad B_k := \mu_k(\iota_k(A)) \subset \Delta_k.$$

Let $\omega_k := \frac{1}{k}\iota_k^* \omega_{\mathrm{FS}}$ be the restriction of the Fubini-Study metric, here we normalize ω_k so that it represents $c_1(L)$. Note that π_k is given by

$$\pi_k : z \longmapsto \frac{1}{\sum |s_i(z)|^2} \left(|s_1(z)|^2_{h_0}, \ldots, |s_{k^n}(z)|^2_{h_0}\right).$$

From a property of theta functions, π_k is invariant under the translations

$$z = \Omega x + y \longmapsto \Omega(x + a) + y, \quad a \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n.$$

in the fiber direction by lattice points of order k. Hence π_k looks close to π for large k.

Note that $\pi_k : A \to B_k$ can not be a Lagrangian fibration since $\dim_{\mathbb{R}} B_k = \dim_{\mathbb{R}} A = 2n$. We thus compare $\pi : (A, \omega_0) \to T^n$ and $\pi_k : (A, \omega_k) \to B_k$ as maps between metric spaces. For that purpose, we need to define distances on T^n and B_k. We define a metric on T^n in such a way that $\pi : (A, \omega_0) \to T^n$ is a Riemannian submersion. The distance on B_k is induced from a metric on the moment polytope Δ_k. A metric on Δ_k is also defined in such a way that

$$\mu_k : \left(\mathbb{CP}^{N_k}, \frac{1}{k}\omega_{\mathrm{FS}}\right) \longrightarrow \Delta_k$$

is a Riemannian submersion in the interior of Δ_k.

Theorem 4.3 (N. [21]). *The sequence of maps $\pi_k : (A, \omega_k) \to B_k$ converges to $\pi : (A, \omega) \to T^n$ in the following sense.*

(i) $\{\omega_k\}$ converges to ω in the C^∞-topology as $k \to \infty$. In particular, the sequence $\{(A, \omega_k)\}$ of Riemannian manifolds converges to (A, ω_0) with respect to the Gromov-Hausdorff distance.

(ii) $\{B_k\}$ converge to T^n as $k \to \infty$ with respect to the Gromov-Hausdorff distance.

(iii) $\{\pi_k\}$ converges to π as maps between metric spaces (see [22] for the definition).

Outline of the proof

Recall that

$$\omega_k - \omega_0 = \sqrt{-1}\partial\bar{\partial}\log\left(\sum_i \|s_i\|_{h_0}\right) = \sqrt{-1}\partial\bar{\partial}\log \Pi_k(z, z).$$

Hence (i) follows from the following theorem.

Theorem 4.4 (Ruan [24], Zelditch [34]). *Let (X, ω) be a compact Kähler manifold and $(L, h) \to X$ a Hermitian line bundle such that $\omega = c_1(L, h)$. Then, for each q, there exists a constant $C_q > 0$ independent of k such that*

$$\|\Pi_k(z, z) - k^n\|_{C^q} \leq C_q k^{n-1}\,.$$

For the proof of (ii), we decompose $T\mathbb{CP}^{N_k}$ into the horizontal and vertical components:

$$\begin{array}{ccccc} T_p\mathbb{CP}^{N_k} & = & T_{\mathbb{CP}^{N_k}/\Delta_k,p} & \oplus & (T_{\mathbb{CP}^{N_k}/\Delta_k,p})^\perp \\ \xi & = & \xi^V & + & \xi^H \end{array} \tag{12}$$

where $T_{\mathbb{CP}^{N_k}/\Delta_k,p} = \ker d\mu_k$ is the tangent space to the fiber of μ_k and $(T_{\mathbb{CP}^{N_k}/\Delta_k,p})^\perp$ is its orthogonal complement with respect to the Fubini-Study metric. Similarly we decompose the tangent space of A:

$$T_zA = T_{A/T^n,z} \oplus (T_{A/T^n,z})^\perp\,, \tag{13}$$

where $(T_{A/T^n,z})^\perp$ is the orthogonal complement of $T_{A/T^n,z} = \ker d\pi$ with respect to the flat metric ω_0. Then the metrics on Δ_k and T^n are the restrictions of ω_k and ω_0 on the horizontal subspaces, respectively. Since we know from (i) that ω_0 and ω_k are "close" for large k , it suffices to show that also the decompositions (12) and (13) are "close".

Lemma 4.5. *(i) If $\xi \in T_{A/T^n,z}$, then*

$$\left|d\iota_k(\xi)^H\right| \leq \frac{C}{\sqrt{k}}|\xi|\,.$$

(ii) If $\eta \in (T_{A/T^n,z})^{\perp}$, then

$$\left| d\iota_k(\eta)^V \right| \le \frac{C}{\sqrt{k}} |\eta| \,.$$

This lemma follows from the asymptotic behavior of theta functions. By using Lemma 4.5, we have an estimate

$$d_{\mathrm{GH}}(T^n, B_k) \le \frac{C}{\sqrt{k}}$$

for the Gromov-Hausdorff distance between T^n and B_k. In fact, we can show that the composition

$$\varphi_k = \pi_k \circ \sigma_0 : T^n \longrightarrow B_k$$

of the zero section $\sigma_0 : T^n \to A$ and π_k is "almost isometric" (a $(C/\sqrt{k})$-Hausdorff approximation (see [11] for the definition)).

4.2 The case of Kummer varieties

Let (A, L) be a polarized Abelian variety as above. The Kummer variety of A is an orbifold defined by

$$X = A/(-1)_A \,,$$

where $(-1)_A$ is the inverse morphism $z \mapsto -z$. For $n = 2$, X is a singular K3 surface. Since L is symmetric, there exists a line bundle $M \to X$ satisfying

$$p^* M \cong L^2 \,,$$

where $p : A \to X$ is the natural projection. From the fact that $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(A)$ is injective, we have

$$p^* M^k \cong L^{2k} \,.$$

Furthermore, $p^* : H^0(X, M^k) \to H^0(A, L^{2k})$ is injective and the image is spanned by

$$s_{b_i} + s_{-b_i} \,, \quad b_i \in T^n_{2k}$$

(see [5] and [25]). Note that

$$N_k + 1 := \dim H^0(X, M^k) = 2^{n-1}(k^n + 1) \,.$$

Let ω be an orbifold Kähler metric induced from the flat metric $2\omega_0$ on A. Then $[\omega] = c_1(M)$. We have also a Lagrangian fibration

$$\pi : (X, \omega) \to B = T^n/(-1)$$

induced by $\pi : A \to T^n$. We identify $H^0(X, M^k)$ with its image in $H^0(A, L^{2k})$ and set

$$t_i = \begin{cases} \dfrac{1}{\sqrt{2^n}}(s_{b_i} + s_{-b_i}) \,, & \text{if } b_i \in T^n_{2k} \backslash T^n_2, \\ \dfrac{1}{\sqrt{2^{n-1}}} s_{b_i} \,, & \text{if } b_i \in T^n_2 \,. \end{cases}$$

Then $\{t_i\}$ is an orthonormal basis of $H^0(X, M^k)$.

We denote by $\iota_k : X \to \mathbb{CP}^{N_k}$ the projective embedding defined by $\{t_i\}$, $\pi_k : X \to B_k$ the restriction of the moment map of $\mathbb{CP}^{N_k}$, and $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$. Then the same theorem holds for X.

Theorem 4.6. *(i) $\{(X, \omega_k)\}$ converges to (X, ω) with respect to the Gromov-Hausdorff distance.*

(ii) B_k converge to B with respect to the Gromov-Hausdorff distance.

(iii) $\{\pi_k\}$ converges to π as maps between metric spaces.

Outline of the proof

(i) follows from an orbifold version of Theorem 4.4:

Theorem 4.7 (Dai-Liu-Ma [7]). *Let (X, ω) be a compact Kähler orbifold of dimension $n \geq 2$ and $(M, h) \to X$ an orbifold Hermitian line bundle with $c_1(M, h) = \omega$. For $k \gg 1$, we consider a projective embedding $\iota_k : X \to \mathbb{CP}^{N_k}$ defined by an orthonormal basis of $H^0(X, M^k)$, and we put $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ as above. Then*

$$\|\omega - \omega_k\|_{C^q, z} \leq C_q \left(\frac{1}{k} + k^{\frac{q}{2}} e^{-k\delta r(z)^2} \right),$$

where $\| \cdot \|_{C^q,z}$ is the C^q-norm at $z \in X$, and $r(z)$ is the distance between z and the singular set $\mathrm{Sing}\,(X)$ of X.

(ii) Note that each singular fiber is isomorphic to $T^n/(-1)$ and appears on the singular set $\mathrm{Sing}(B) = T_2^n/(-1)$ of B. For each $b \in \mathrm{Sing}(B)$, we denote the $\sqrt{(1/\delta k)\log k}$-neighborhood of the singular fiber $\pi^{-1}(b)$ by

$$N_{b,k} = \left\{ z \in X \;\middle|\; d(z, \pi^{-1}(b)) \leq \sqrt{\frac{\log k}{\delta k}} \right\},$$

where δ is the constant in Theorem 4.7, and set

$$X(k) = X \backslash \bigcup_{b \in \mathrm{Sing(B)}} N_{b,k}\,.$$

Then we can show that $\pi(N_{b,k})$ and $\pi_k(N_{b,k})$ are small for large k (their diameters can be bounded by $O\left(\sqrt{(1/k)\log k}\right)$). Hence the neighborhoods of singular fibers do not affect to the Gromov-Hausdorff convergence. On the other hand, we have the same estimates as in Lemma 4.5 on $X(k)$. Hence we can apply the same arguments to this situation.

4.3 Toward a Generalization

Finally we apply the construction (10) of theta functions to general cases, and study asymptotic behavior of the resulting sections.

Let (X, ω) be a compact Kähler manifold, $(L, h) \to X$ a Hermitian line bundle such that $c_1(L, h) = \omega$. We denote the Bergman kernel of L^k by $\Pi_k(z, w)$. Suppose that X admits a Lagrangian fibration $\pi : X \to B$. We consider only on a neighborhood of a smooth torus fiber. Let $(x^1, \dots, x^n, y^1, \dots, y^n)$ be an action-angle coordinate as in Section 2. Then $\pi^{-1}(b)$ is a k-BS fiber if and only if its action variables $(y^1, \dots, y^n) \in \frac{1}{k}\mathbb{Z}^n$. For each k-BS fiber $\pi^{-1}(b_i)$, we take a covariantly constant section σ_i of $(L^k, \nabla)|_{\pi^{-1}(b_i)}$ with $\|\sigma_i\|_{L^2} = 1$, and define a holomorphic section of L^k by

$$s_i = \left(\frac{k}{\pi}\right)^{-n/4} \int_{\pi^{-1}(b)} \Pi_k(z, x)\sigma_i(x) dvol(x) \in H^0(X, L^k),$$

where $dvol$ is the volume form on $\pi^{-1}(b)$ induced from the Kähler metric. This is the same construction as in [6], with certain half densities. Recall that the key properties for the proof of Theorem 4.3 are

- $\sum_i |s_i(z)|_h^2 = \left(\frac{k}{2\pi}\right)^n + O(k^{n-1})$ (*i.e.* the leading term is constant),
- s_i has a peak along the fiber $\pi^{-1}(b)$.

The second property follows from a fact that the Bergman kernel $\Pi_k(z, w)$ has a peak along the diagonal set of $X \times X$.

Theorem 4.8. *For $z \in X$ in the smooth part π, we have*

$$\sum_i |s_i(z)|_h^2 = k^n \left(\frac{\sqrt{\det g(z)}}{V(z)} + O\left(\frac{(\log k)^3}{\sqrt{k}}\right)\right)$$

in the C^0-topology, where $dvol(z) = \sqrt{\det g(z)} dx^1 \wedge \cdots \wedge dx^n$, and $V(z)$ is the volume of the fiber $\pi^{-1}(\pi(z))$. In particular, if $dvol$ is invariant under the Hamiltonian flows of y^i's, then

$$\sum_i |s_i(z)|_h^2 = k^n \left(\frac{1}{(2\pi)^n} + O\left(\frac{(\log k)^3}{\sqrt{k}}\right)\right)$$

in the C^0-topology.

References

[1] J. E. Andersen, *Geometric quantization of symplectic manifolds with respect to reducible non-negative polarizations*, Comm. Math. Phys. 183 (1997), 401–421.

[2] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces* Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523–615.

[3] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. 14 1961 187–214.

[4] V. Bargmann, *Remarks on a Hilbert space of analytic functions* Proc. Nat. Acad. Sci. U.S.A. 48 1962 199–204.

[5] Th. Bauer, *Projective images of Kummer surfaces*, Math. Ann. 299 (1994), 155–170.

[6] D. Borthwick, T. Paul, and A. Uribe, *Legendrian distributions with applications to relative Poincaré series*, Invent. Math. 122 (1995), 359–402.

[7] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, C. R. Math. Acad. Sci. Paris 339 (2004), no. 3, 193–198.

[8] I. V. Dolgachev, *Mirror symmetry for lattice polarized $K3$ surfaces*, Algebraic geometry, 4. J. Math. Sci. 81 (1996), no. 3, 2599–2630.

[9] S. K. Donaldson, *Planck's constant in complex and almost-complex geometry*, XIIIth International Congress on Mathematical Physics (London, 2000), 63–72, Int. Press, Boston, MA, 2001

[10] J. I. Duistermaat, *On global action-angle coordinates*, Comm. Pure. Appl. Math. Vol.XXXIII (1980), 687–706.

[11] K. Fukaya, *Hausdorff convergence of Riemannian manifolds and its applications*, Recent topics in differential and analytic geometry, 143–238, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, 1990.

[12] K. Fukaya, *Mirror symmetry of Abelian varieties and multi-theta functions*, J. Alg. Geom. 11 (2002), 393–512.

[13] I. M. Gel′fand and M. L. Cetlin, *Finite-dimensional representations of the group of unimodular matrices*, (Russian) Doklady Akad. Nauk SSSR (N.S.) 71, (1950) 825–828.

[14] M. Gross, *Special Lagrangian fibrations I: Topology*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 156–193, World Sci. Publishing, River Edge, NJ, 1998.

[15] M. Gross, *Special Lagrangian fibrations II: Geometry*, A survey of techniques in the study of special Lagrangian fibrations. Surveys in differential geometry: differential geometry inspired by string theory, 341–403, Surv. Differ. Geom., 5, Int. Press, Boston, MA, 1999.

[16] V. Guillemin and S. Sternberg, *The Gelfand-Cetlin system and quantization of the complex flag manifolds*, J. Funct. Analysis 52 (1983), 106–128.

[17] B. C. Hall, *The Segal-Bergmann coherent state transform for compact Lie groups*, J. Funct. Anal. 122 (1994), 103–151.

[18] N. J. Hitchin, *Flat connections and geometric quantization*, Comm. Math. Phys. 131 (1990), no. 2, 347–380.

[19] L. C. Jeffrey and J. Weitsman, *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde formula*, Comm. Math. Phys. 150 (1992), 593–630.

[20] D. Mumford, *Tata lectures on theta III*, Progress in mathematics, 97, Birkhäuser (1991).

[21] Y. Nohara, *Projective embeddings and Lagrangian fibrations of Abelian varieties*, Math. Ann. 333 (2005), 741–757.

[22] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, 171. Springer-Verlag, New York, 1998.

[23] A. Polishchuk and E. Zaslow, *Categorical mirror symmetry: the elliptic curve*, Adv. Theor. Math. Phys. 2 (1998), 443–470, math.AG/9801119.

[24] W.-D. Ruan, *Canonical coordinates and Bergman metrics*, Comm. Anal. Geom. 6 (1998), no. 3, 589–631.

[25] R. Sasaki, *Bounds on the degree of the equations defining Kummer varieties*, J. Math. Soc. Japan 33 (1981), no. 2, 323–333.

[26] I. E. Segal, *Mathematical problems of relativistic physics*, Lectures in Applied Mathematics (proceedings of the Summer Seminar, Boulder, Colorado, 1960), Vol. II American Mathematical Society, Providence, R.I. 1963.

[27] I. E. Segal, *Mathematical characterization of the physical vacuum for a linear Bose-Einstein field*, Illinois J. Math. 6 1962 500–523.

[28] I. E. Segal, *The complex-wave representation of the free boson field* in "Topics in functional analysis" (essays dedicated to M. G. Kreĭn on the occasion of his 70th birthday), Adv. in Math. Suppl. Stud., 3, Academic Press, New York-London, 1978 321–343.

[29] J. Śniatycki, *On cohomology appearing in geometric quantization*, Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), pp. 46–66. Lecture Notes in Math., Vol. 570, Springer, Berlin, 1977.

[30] A. Strominger, S. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. B, 479 (1996), 243–259.

[31] A. Tyurin, *Geometric quantization and mirror symmetry*, math.AG/9902027.

[32] A. Tyurin, *Quantization, classical and quantum field theory and theta functions*, CRM Monograph Series, 21. American Mathematical Society, Providence, RI, 2003.

[33] J. Weitsman, Quantization via real polarization of the moduli space of flat connections and Chern-Simons gauge theory in genus one, Comm. Math. Phys. 137 (1991), 175–190.

[34] S. Zelditch, *Szegö kernels and a theorem of Tian*, International Math. Res. Notices 1998, No.6, 317–331.

[35] D. P. Zelobenko, *Compact Lie groups and their representations*, Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, Providence, R.I., 1973.